Math-II

Definition 0.1 Coset of vector space: *Let V be a vector space and let W be a subspace of V then for $v \in V$, Coset of W in V is $\{v + w | w \in W\}$ and it is denoted by $v + W$.*
$v + W = \{v + w | w \in W\}$

Proposition 0.2 *Let W be a subspace of a real vector space V. For $v_1; v_2 \in V$, show that*

(i) $v + W = W$ *iff* $v \in W$

(ii) $v_1 + W = v_2 + W$ *iff* $v_1 - v_2 \in W$

(ii) either $v_1 + W \cap v_2 + W = \phi$ *or* $v_1 + W = v_2 + W$

(i) $v + W = W$ iff $v \in W$
Proof: Suppose $v + W = W$
T.P.T $v \in W$
$v = v + 0 \in v + W = \in W$
$\implies v \in W$
Conversely: Suppose $v \in W$
T.P.T $v + W = W$
Consider any $x \in v + W$
$\implies x = v + w$, for some $w \in W$
$\implies x = v + w \in W$ since W is subspace of V.
$\implies x \in W$
$\implies v + W \subset W \cdots\cdots\cdots\cdots (i)$
Consider any $y \in W$
Since, $v \in W \implies (-v) \in W$, since W is subspace of V.
Therefore, $y = v + (-v) + y \in v + W$
$\implies W \subset v + W. \cdots\cdots\cdots\cdots (ii)$
By (i) & (ii)
$\implies v + W = W$

(ii) $v_1 + W = v_2 + W$ iff $v_1 - v_2 \in W$ **Proof:** Suppose $v_1 + W = v_2 + W$
T.P.T $v_1 - v_2 \in W$
Let $x \in v_1 + W = v_2 + W$
$\implies x \in v_1 + W$ and $x \in v_2 + W$
$\implies x = v_1 + w_1$ and $x = v_2 + w_2$ for some $w_1, w_2 \in W$
$\implies v_1 + w_1 x = v_2 + w_2$
$\implies v_1 - v_2 = w_2 - w_1 \in W$ since W is subspace of V
$\implies v_1 - v_2 \in W$
Conversly: Suppose $v_1 - v_2 \in W$
T.P.T $v_1 + W = v_2 + W$
Consider any $x \in v_1 + W$
$\implies x = v_1 + w_1$ for some $w_1 \in W$

$\implies x = v_1 + w_1 = v_2 - v_2 + v_1 + w_1$
$\implies x = v_2 + v_1 - v_2 + w_1 \in v_2 + W$, since W is subspace of $V \implies v_1 - v_2 + w_1 \in +W$
$\implies x \in v_2 + W$
$\implies v_1 + W \subset v_2 + W \cdots\cdots\cdots\cdots (i)$
Consider any $x \in v_2 + W$
$\implies y = v_2 + w_2$ for some $w_2 \in W$
$\implies y = v_2 + w_2 = v_1 - v_1 + v_2 + w_2$
$\implies y = v_1 + v_2 - v_1 + w_2 \in v_2 + W$, since W is subspace of $V \implies v_2 - v_1 + w_2 \in +W$
$\implies y \in v_2 + W$
$\implies v_2 + W \subset v_1 + W \cdots\cdots\cdots\cdots (ii)$
From (i) and (ii)
$v_1 + W = v_2 + W$

Theorem 0.3 *Let W be a subspace of real vector space V and $V/W = \{v + W | v \in W\}$ then addition defined by $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$ and scalar multiplication defined by $\alpha \cdot (v + W) = \alpha \cdot v + W$ are well defined. Further show that V/W under these operation is real vector space.*

Definition 0.4 Quotient Space: *Let W be a subspace of real vector space V . V/W is vector space under addition and scalar multiplication then $(V/W, +, \cdot)$ is called Quotient Space.*

Proposition 0.5 *Let V be a finite dimension real vector space and let W be subspace of V then show that dim(V/W) = dim(V) - dim(W)*

Proof: Let V be a finite dimension real vector space
Let dim$V = n$, hence W is finite dimension and dim$W = m \leq n$
Let $S = \{w_1, w_2, \cdots, w_m\}$ is basis of W
$\implies S$ is linearly independent set in V
S can be extended to $\{w_1, w_2, \cdots, w_m, v_1, v_2, \cdots, v_{n-m}\}$ to basis of V.
Let $S_1 = \{w_1, w_2, \cdots, w_m, v_1, v_2, \cdots, v_{n-m}\}$ be basis of V.
T.P.T dim(V/W) = dim(V) - dim$(W) = n - m$
Let $B = \{v_1 + W, v_2 + W, \cdots, v_{n-m} + W\}$
T.P.T B is basis of V/W.
$a_1(v_1 + W) + a_2(v_2 + W) + \cdots + a_{n-m}(v_{n-m} + W) = 0 + W$, for some $a_i \in \mathbb{R}$
$a_1v_1 + W + a_2v_2 + W + \cdots + a_{n-m}v_{n-m} + W = 0 + W$
$a_1v_1 + a_2v_2 + \cdots + a_{n-m}v_{n-m} + W = 0 + W$
$a_1v_1 + a_2v_2 + \cdots + a_{n-m}v_{n-m} \in W$
Since, S is basis of $W \implies L(S) = W$.
$a_1v_1 + a_2v_2 + \cdots + a_{n-m}v_{n-m} = b_1w_1 + b_2w_2 + \cdots + b_mw_m$, for some $b_i \in \mathbb{R}$
$b_1w_1 + b_2w_2 + \cdots + b_mw_m - a_1v_1 - a_2v_2 + \cdots - a_{n-m}v_{n-m} = 0$
$b_1w_1 + b_2w_2 + \cdots + b_mw_m + (-a_1)v_1 + (-a_2)v_2 + \cdots + (-a_{n-m})v_{n-m} = 0$
since, $S_1 = \{w_1, w_2, \cdots, w_m, v_1, v_2, \cdots, v_{n-m}\}$ be basis of V
$\implies S_1$ is Linearly independent.
$\implies b_1 = b_2 = \cdots = b_m = -a_1 = -a_2 = \cdots = -a_{n-m} = 0$

$\implies a_1 = a_2 = \cdots = a_{n-m} = 0$
$\implies B$ is linearly independent.
T.P.T $V/W = L(W)$
consider any $v + W \in V/W$, for some $v \in V$
So, $v \in V$
Since, S_1 is basis of $V \implies L(S) = V$.
$\implies v = a_1w_1 + a_2w_2 + \cdots + a_mw_m + b_1v_1 + b_2v_2 + \cdots + b_{n-m}v_{n-m}$
$\implies v + W = a_1w_1 + a_2w_2 + \cdots + a_mw_m + b_1v_1 + b_2v_2 + \cdots + b_{n-m}v_{n-m} + W$
$\implies v + W = v' + b_1v_1 + b_2v_2 + \cdots + b_{n-m}v_{n-m} + W$ where $v' = a_1w_1 + a_2w_2 + \cdots + a_mw_m$
$\implies v + W = b_1v_1 + b_2v_2 + \cdots + b_{n-m}v_{n-m} + v' + W$
Since, $v' = a_1w_1 + a_2w_2 + \cdots + a_mw_m \in L(S) = W$
$v' + W = W$
$\implies \implies v + W = b_1v_1 + b_2v_2 + \cdots + b_{n-m}v_{n-m} + W$
$\implies v + W = b_1v_1 + W + b_2v_2 + W + \cdots + b_{n-m}v_{n-m} + W$
$\implies v + W = b_1(v_1 + W) + b_2(v_2 + W) + \cdots + b_{n-m}(v_{n-m} + W)$
$\implies v + W \in L(B)$
$\implies V/W \subset L(B) \cdots\cdots\cdots\cdots (i)$
Clearly, $L(B) \subset V/W \cdots\cdots\cdots\cdots (ii)$
From (i) and (ii)
$\implies V/W = L(B)$
Therefore, B is basis of V/W.
Hence, $\dim(V/W) = n - m =$ dim(V) - dim(W).

Theorem 0.6 *State and prove the 'First Isomorphism Theorem of vector space' (Fundamental theorem of vector space homomorphism).*

Statement: Let V, W be real vector spaces. Let $T : V \to W$ be an onto linear transformation with $K = \text{ket}T$ then V/K is isomorphic to W.
Proof: Let V, W be real vector spaces. Let $T : V \to W$ be an onto linear transformation with $K = \text{ket}T$.
$K = \ker T = \{x \in V | T(x) = 0\}$

$$V/K = \{x + K \mid x \in V\}$$

We defined $\phi : V/K \to W$ as $\phi(x + K) = T(x)$
we prove:
(i) ϕ is well defined
(ii) ϕ is linear transformation.
(iii) ϕ is onto
(iv) ϕ is ono-one
(i) T.P.T ϕ is well defined
Consider $x + K = y + K$ in V/K
$\implies x - y + K = K$
$\implies x - y \in K = \ker T$

$\implies T(x-y)=0$
$\implies T(x)-T(y)=0$
$\implies T(x)=T(y)$
$\implies \phi(x+K)=\phi(y+K)$
(ii) T.P.T. ϕ is linear transformation.
Consider any $x+K, y+K \in V/K$ and $a, b \in \mathbb{R}$

$$\begin{aligned}
\phi[a(x+K)+b(y+K)] &= \phi[(ax+K)+(by+K)] \\
&= \phi[(ax+by)+K] \\
&= T(ax+by) \\
&= aT(x)+bT(y) \\
&= a\phi(x+K)+b\phi(y+K)
\end{aligned}$$

$\implies \phi$ is linear transformation
(iii) T.P.T ϕ is onto
Consider any $y \in W$
Since, $T: V \to W$ be an onto linear transformation
$\implies \exists x \in V$
$\implies T(x)=y$
Let $x+K \in V/K$, Then
$\implies \phi(x+K)=T(x)=y$
$\implies \phi$ is onto
(iv) T.P.T ϕ is ono-one
Consider, $\phi(x+K)=\phi(y+K)$
$\implies T(x)=T(y)$
$\implies T(x)-T(y)=0$
$\implies T(x-y)=0$
$\implies x-y \in =\ker T=K$
$\implies x-y+K=K$
$\implies x+K=y+K$
$\implies \phi$ is ono-one
From (i) to (iv)
$\implies \phi$ is linear isomorphism
$\implies V/K$ is isomorphic to W.

Proposition 0.7 *Let V be a finite dimensional inner product vector space and $T: V \to V$ be a linear transformation then prove that T is orthogonal linear transformation iff $\|T(X)\| = \|(X)\|,\ \forall X \in V$.*

Proof: Let V be a
nite dimensional inner product vector space and $T: V \to V$ be linear transformation
Suppose T is orthogonal linear transformation
T.P.T $\|T(X)\| = \|(X)\|,\ \forall X \in V$

T is orthogonal linear transformation $\implies \langle T(X), T(Y)\rangle = \langle X, Y\rangle, \forall X, Y \in V$
In particular, $X = Y$ then
$\implies \langle T(X), T(X)\rangle = \langle X, X\rangle$
$\implies \|T(X)\|^2 = \|(X)\|^2$
$\implies \|T(X)\| = \|(X)\|, \ \forall\ X \in V.$
Conversely: Suppose $\implies \|T(X)\| = \|(X)\|, \ \forall\ X \in V.$
T.P.T T is orthogonal linear transformation
Consider any $X, Y \in V$
T.P.T $\langle T(X), T(Y)\rangle = \langle X, Y\rangle$
$\|T(X+Y)\| = \|(X+Y)\|$
$\implies \|T(X) + T(Y)\|^2 = \|X+Y\|^2$

$$\begin{aligned}
\text{L.H.S} &= \|T(X)+T(Y)\|^2 \\
&= \langle T(X)+T(Y), T(X)+T(Y)\rangle \\
&= \langle T(X), T(X)\rangle + \langle T(X), T(Y)\rangle + \langle T(Y), T(X)\rangle + \langle T(Y), T(Y)\rangle \\
&= \langle T(X), T(X)\rangle + \langle T(X), T(Y)\rangle + \langle T(X), T(Y)\rangle + \langle T(Y), T(Y)\rangle \\
&= \|T(X)\|^2 + 2\langle T(X), T(Y)\rangle + \|T(Y)\|^2 \\
&= \|X\|^2 + 2\langle T(X), T(Y)\rangle + \|Y\|^2, \qquad \text{since } \|T(X)\| = \|X\|, \|T(Y)\| = \|Y\| \\
\text{R.H.S} &= \|X+Y\|^2 \\
&= \langle X+Y, X+Y\rangle \\
&= \langle X, X\rangle + \langle X, Y\rangle + \langle Y, X\rangle + \langle Y, Y\rangle \\
&= \langle X, X\rangle + \langle X, Y\rangle + \langle X, Y\rangle + \langle Y, Y\rangle \\
&= \|X\|^2 + 2\langle X, Y\rangle + \|Y\|^2
\end{aligned}$$

$\|T(X) + T(Y)\|^2 = \|X+Y\|^2$
$\implies \|T(X)\|^2 + 2\langle T(X), T(Y)\rangle + \|T(Y)\|^2 = \|X\|^2 + 2\langle X, Y\rangle + \|Y\|^2$
$\implies 2\langle T(X), T(Y)\rangle = 2\langle X, Y\rangle$
$\implies \langle T(X), T(Y)\rangle = \langle X, Y\rangle, \ \forall\ X, Y \in V$
$\implies T$ is orthogonal linear transformation

Proposition 0.8 *Let V be a finite dimensional inner product vector space and $T : V \to V$ be a linear transformation then prove that T is orthogonal linear transformation iff If $\{e_1, e_2, \ldots e_n\}$ is orthonormal basis of V then $\{T(e_1), T(e_2), \ldots, T(e_n)\}$ is orthonormal basis of V*

Let V be a finite dimensional inner product vector space and $T : V \to V$ be a linear transformation
Suppose If $\{e_1, e_2, \ldots e_n\}$ is orthonormal basis of V then, $\{T(e_1), T(e_2), \ldots, T(e_n)\}$ is orthonormal basis of V
T.P.T T is orthogonal linear transformation
Consider any $X, Y \in V$

T.P.T $\langle T(X),T(Y)\rangle = \langle X,Y\rangle\,,\ \forall\ X,Y\in V$
Since, $\{e_1,e_2,\dots e_n\}$ is basis of V
$\implies X = a_1e_1+a_2e_2+\cdots+a_ne_n$ and $Y = b_1e_1+b_2e_2+\cdots+b_ne_n$, for some $a_i, b_j \in \mathbb{R}$
Consider,
$\langle X,Y\rangle$
$= \langle a_1e_1+a_2e_2+\cdots+a_ne_n, b_1e_1+b_2e_2+\cdots+b_ne_n\rangle$
$= \langle a_1e_1,b_1e_1\rangle+\langle a_1e_1,b_2e_2\rangle+\cdots+\langle a_1e_1,b_ne_n\rangle+\langle a_2e_2,b_1e_1\rangle+\langle a_2e_2,b_2e_2\rangle+\cdots+\langle a_2e_2,b_ne_n\rangle+\cdots+\langle a_ne_n,b_1e_1\rangle+\langle a_ne_n,b_2e_2\rangle+\cdots+\langle a_ne_n,b_ne_n\rangle$
$= a_1b_1\langle e_1,e_1\rangle+a_1b_2\langle e_1,e_2\rangle+\cdots+a_1b_n\langle e_1,e_n\rangle+a_2b_1\langle e_2,e_1\rangle+a_2b_2\langle e_2,e_2\rangle+\cdots+a_2b_n\langle e_2,e_n\rangle+\cdots+a_nb_1\langle e_n,e_1\rangle+a_nb_2\langle e_n,e_2\rangle+\cdots+a_nb_n\langle e_n,e_n\rangle$

$$\begin{aligned}\text{Since, } \langle e_i,e_j\rangle &= 1, \qquad \text{if } i=j\\ &= 0 \qquad \text{if } i\neq j\end{aligned}$$

$\langle X,Y\rangle = a_1b_1(1)+a_1b_2(0)+\cdots+a_1b_n(0)+a_2b_1(0)+a_2b_2(1)+\cdots+a_2b_n(0)+\cdots+a_nb_1(0)+a_nb_2(0)+\cdots+a_nb_n(1)$
$= a_1b_1+a_2b_2+\cdots+a_nb_n$
Now,
$T(X) = T(a_1e_1+a_2e_2+\cdots+a_ne_n) = a_1T(e_1)+a_2T(e_2)+\cdots+a_nT(e_n)$
$T(Y) = T(b_1e_1+b_2e_2+\cdots+b_ne_n) = b_1T(e_1)+b_2T(e_2)+\cdots+b_nT(e_n)$
Consider,
$\langle T(X),T(Y)\rangle$
$= \langle a_1T(e_1)+a_2T(e_2)+\cdots+a_nT(e_n), b_1T(e_1)+b_2T(e_2)+\cdots+b_nT(e_n)\rangle$
$= \langle a_1T(e_1),b_1T(e_1)\rangle+\langle a_1T(e_1),b_2T(e_2)\rangle+\cdots+\langle a_1T(e_1),b_nT(e_n)\rangle+\langle a_2T(e_2),b_1T(e_1)\rangle+\langle a_2T(e_2),b_2T(e_2)\rangle+\cdots+\langle a_2T(e_2),b_nT(e_n)\rangle+\cdots+\langle a_nT(e_n),b_1T(e_1)\rangle+\langle a_nT(e_n),b_2T(e_2)\rangle+\cdots+\langle a_nT(e_n),b_nT(e_n)\rangle$
$= a_1b_1\langle T(e_1),T(e_1)\rangle+a_1b_2\langle T(e_1),T(e_2)\rangle+\cdots+a_1b_n\langle T(e_1),T(e_n)\rangle+a_2b_1\langle T(e_2),T(e_1)\rangle+a_2b_2\langle T(e_2),T(e_2)\rangle+\cdots+a_2b_n\langle T(e_2),T(e_n)\rangle+\cdots+a_nb_1\langle T(e_n),T(e_1)\rangle+a_nb_2\langle T(e_n),T(e_2)\rangle+\cdots+a_nb_n\langle T(e_n),T(e_n)\rangle$

$$\begin{aligned}\text{Since, } \langle T(e_i),T(e_j)\rangle &= 1, \qquad \text{if } i=j\\ &= 0 \qquad \text{if } i\neq j\end{aligned}$$

$\langle T(X),T(Y)\rangle = a_1b_1(1)+a_1b_2(0)+\cdots+a_1b_n(0)+a_2b_1(0)+a_2b_2(1)+\cdots+a_2b_n(0)+\cdots+a_nb_1(0)+a_nb_2(0)+\cdots+a_nb_n(1)$
$= a_1b_1+a_2b_2+\cdots+a_nb_n$
$\implies \langle T(X),T(Y)\rangle = a_1b_1+a_2b_2+\cdots+a_nb_n = \langle X,Y\rangle$
$\implies \langle T(X),T(Y)\rangle = \langle X,Y\rangle$
Hence, T is orthogonal linear transformation.
conversely: Suppose T is orthogonal linear transformation.
Let $\{e_1,e_2,\dots e_n\}$ be orthonormal basis of V
$\implies \dim V = n$

T.P.T $\{T(e_1), T(e_2), \dots, T(e_n)\}$ is orthonormal basis of V
T is orthogonal linear transformation $\implies \langle T(X), T(Y) \rangle = \langle X, Y \rangle, \ \forall \ X, Y \in V$
Therefore,

$$\begin{aligned} \langle T(e_i), T(e_j) \rangle = \langle e_i, e_j \rangle &= 1, \qquad \text{if } i = j \\ &= 0 \qquad \text{if } i \neq j \end{aligned}$$

$\implies \{T(e_1), T(e_2), \dots, T(e_n)\}$ is orthonormal set in $V \cdots\cdots (i)$
Consider, $a_1 T(e_1) + a_2 T(e_2) + \dots + a_n T(e_n) = 0$, for some $a_i \in \mathbb{R}$
Now, $\langle T(e_i), a_1 T(e_1) + a_2 T(e_2) + \dots + a_n T(e_n) \rangle = \langle T(e_i), 0 \rangle$
$\implies \langle T(e_i), T(e_1) \rangle + a_2 \langle T(e_1), T(e_i) \rangle + \cdots a_i \langle T(e_i), T(e_i) \rangle + \cdots + a_n \langle T(e_i), T(e_n) \rangle = 0$
$\implies a_1(0) + a_2(0) + \dots + a_i(1) + \dots + a_n(0) = 0$
$\implies a_i = 0, \ \forall \ i = 1, 2, \dots n$
$\implies a_1 = a_2 = \dots = a_n = 0$
$\implies \{T(e_1), T(e_2), \dots, T(e_n)\}$ is linearly independent set in V
As $\dim V = n$
$\implies$ Any linearly independent set with n element form basis of V.
$\implies \{T(e_1), T(e_2), \dots, T(e_n)\}$ is basis of $V \cdots\cdots (ii)$
By (i) and (ii)
Hence, $\{T(e_1), T(e_2), \dots, T(e_n)\}$ is orthonormal basis of V

Theorem 0.9 *Let V be a finite dimensional inner product vector space. If $f : V \to V$ is a function such that $(i) f(0) = 0$ $(ii) \|f(X) - f(Y)\| = \|X - Y\|, \ \forall \ X, Y \in V$ then show that f is an orthogonal linear transformation.*

Proof: Let V be a finite dimensional inner product vector space. If $f : V \to V$ is a function such that $(i) f(0) = 0$ $(ii) \|f(X) - f(Y)\| = \|X - Y\|, \ \forall \ X, Y \in V$
T.P.T f is an orthogonal linear transformation
$(i) f(0) = 0$, Then
$\|f(X) - f(0)\| = \|X - 0\|$
$\|f(X) - 0\| = \|X\|$
$\implies \|f(X)\| = \|X\|, \ \forall \ X \in V$
$(ii) \|f(X) - f(Y)\| = \|X - Y\|$
$\implies \|f(X) - f(Y)\|^2 = \|X - Y\|^2$

Consider,

$$\begin{aligned}
\|f(X)-f(Y)\|^2 &= \langle f(X)-f(Y), f(X)-f(Y)\rangle \\
&= \langle f(X), f(X)\rangle - \langle f(X), f(Y)\rangle - \langle f(Y), f(X)\rangle + \langle f(Y), f(Y)\rangle \\
&= \langle f(X), f(X)\rangle - \langle f(X), f(Y)\rangle - \langle f(X), f(Y)\rangle + \langle f(Y), f(Y)\rangle \\
&= \|f(X)\|^2 - 2\langle f(X), f(Y)\rangle + \|f(Y)\|^2 \\
&= \|X\|^2 - 2\langle f(X), f(Y)\rangle + \|Y\|^2, \qquad \text{since } \|f(X)\| = \|X\|, \|f(Y)\| = \|Y\| \\
\|X+Y\|^2 &= \langle X-Y, X-Y\rangle \\
&= \langle X, X\rangle - \langle X, Y\rangle - \langle Y, X\rangle + \langle Y, Y\rangle \\
&= \langle X, X\rangle - \langle X, Y\rangle - \langle X, Y\rangle + \langle Y, Y\rangle \\
&= \|X\|^2 - 2\langle X, Y\rangle + \|Y\|^2
\end{aligned}$$

$\implies \|X\|^2 - 2\langle f(X), f(Y)\rangle + \|Y\|^2 = \|X\|^2 - 2\langle X, Y\rangle + \|Y\|^2$
$\implies -2\langle f(X), f(Y)\rangle = -2\langle X, Y\rangle$
$\implies \langle f(X), f(Y)\rangle = \langle X, Y\rangle \cdots\cdots\cdots\cdots (I)$
T.P.T T is linear transformation
Let $\{e_1, e_2, \ldots, e_n\}$ be a orthonormal basis of V.

$$\begin{aligned}
\langle f(e_i), f(e_j)\rangle = \langle e_i, e_j\rangle &= 1, \qquad \text{if } i = j \\
&= 0 \qquad \text{if } i \neq j
\end{aligned}$$

Therefore, $\{f(e_1), f(e_2), \ldots, f(e_n)\}$ be a orthonormal basis of V. Consider any $x \in V \implies$ $x = a_1e_1 + a_2e_2 + \cdots + a_ne_n$, for some $a_i \in \mathbb{R}$
Consider $\langle x, e_i\rangle = \langle a_1e_1 + a_2e_2 + \cdots + a_ne_n, e_i\rangle$
$= \langle a_1e_1, e_i\rangle + \langle a_2e_1, e_i\rangle + \cdots + \langle a_ie_i, e_i\rangle + \cdots + \langle a_ne_n, e_i\rangle$
$= a_1\langle e_1, e_i\rangle + a_2\langle e_1, e_i\rangle + \cdots + a_i\langle e_i, e_i\rangle + \cdots + a_n\langle e_n, e_i\rangle$
$= a_1(0) + a_2(0) + \cdots + a_i(1) + \cdot + a_n(0)$
$= a_i$
$a_i = \langle x, e_i\rangle, \ 1 \leq i \leq n$
$\implies x = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \cdots + \langle x, e_n\rangle e_n$
Now, $x \in V \implies f(x) \in V$
$\implies f(x) = \langle f(x), f(e_1)\rangle f(e_1) + \langle f(x), f(e_2)\rangle f(e_2) + \cdots + \langle f(x), f(e_n)\rangle f(e_n)$
$= \langle x, e_1\rangle f(e_1) + \langle x, e_2\rangle f(e_2) + \cdots + \langle x, e_n\rangle f(e_n)$
$= a_1f(e_1) + a_2f(e_2) + \cdots + a_nf(e_n)$
$= \sum_{i=1}^{n} a_if(e_i)$
$f(x) = \sum_{i=1}^{n} a_if(e_i)$
Consider any $x, y \in V$ and $a, b \in \mathbb{R}$
T.P.T $f(ax+by) = af(x) + bf(y)$
$x = \sum_{i=1}^{n} a_ie_i$, $y = \sum_{i=1}^{n} b_ie_i$ for some $a_i, b_j \in \mathbb{R}$.
$\implies f(x) = \sum_{i=1}^{n} a_if(e_i), f(y) = \sum_{i=1}^{n} b_if(e_i)$
Consider $f(ax+by)$
$= f(a\sum_{i=1}^{n} a_ie_i + b\sum_{i=1}^{n} b_ie_i)$

$= f(\sum_{i=1}^{n} aa_i e_i + \sum_{i=1}^{n} bb_i e_i)$
$= f(\sum_{i=1}^{n}(aa_i + bb_i)e_i)$
$= \sum_{i=1}^{n}(aa_i + bb_i)f(e_i)$
$= \sum_{i=1}^{n} aa_i f(e_i) + \sum_{i=1}^{n} bb_i f(e_i)$
$= a\sum_{i=1}^{n} a_i f(e_i) + b\sum_{i=1}^{n} b_i f(e_i)$
$= af(x) + bf(y)$
$\implies f$ is linear transformation $\cdots\cdots\cdots\cdots (II)$
By (I) and $(II) \implies f$ is orthogonal linear transformation

Corollary 0.10 *Let V be a finite dimensional inner product vector space and $f : V \to V$ be an isometry, then show that there exists unique $x_0 \in V$ and an unique orthogonal linear transformation $T : V \to V$ such that $f = L_{x_0}OT$ where $L_{x_0} : V \to V$ is a translation map denoted as $L_{x_0}(X) = X + X_0$.*

Proof: Let V be a finite dimensional inner product vector space and $f : V \to V$ be an isometry
$\implies \|f(X) - f(Y)\| = \|X - Y\|, \ \forall\ X, Y \in V.$
Let $f(0) = x_0$
Let $L_{x_0} : V \to V$ be a translation map defined as $L_{x_0}(X) = X + X_0$ Define $T : V \to V$ as $T(x) = f(x) - x_0$
So, $T(0) = f(0) - x_0 = x_0 - x_0 = 0$ and
$\|T(X) - T(Y)\| = \|f(X) - x_0 - f(Y) + x_0\| = \|f(X) - f(Y)\| = \|X - Y\|.$
we have, $(i)T(0) = 0$ and $(ii)\ \|T(X) - T(Y)\| = \|X - Y\|$ T.P.T T is an orthogonal linear transformation
$(i)T(0) = 0$, Then
$\|T(X) - T(0)\| = \|X - 0\|$
$\|T(X) - 0\| = \|X\|$
$\implies \|T(X)\| = \|X\|, \ \forall\ X \in V$
$(ii)\ \|T(X) - T(Y)\| = \|X - Y\|$
$\implies \|T(X) - T(Y)\|^2 = \|X - Y\|^2$
Consider,

$$\begin{aligned}
\|T(X) - T(Y)\|^2 &= \langle T(X) - T(Y), T(X) - T(Y)\rangle \\
&= \langle T(X), T(X)\rangle - \langle T(X), T(Y)\rangle - \langle T(Y), T(X)\rangle + \langle T(Y), T(Y)\rangle \\
&= \langle T(X), T(X)\rangle - \langle T(X), T(Y)\rangle - \langle T(X), T(Y)\rangle + \langle f(Y), f(Y)\rangle \\
&= \|T(X)\|^2 - 2\langle T(X), T(Y)\rangle + \|T(Y)\|^2 \\
&= \|X\|^2 - 2\langle T(X), T(Y)\rangle + \|Y\|^2, \qquad \text{since } \|T(X)\| = \|X\|, \|T(Y)\| = \|Y\| \\
\|X - Y\|^2 &= \langle X - Y, X - Y\rangle \\
&= \langle X, X\rangle - \langle X, Y\rangle - \langle Y, X\rangle + \langle Y, Y\rangle \\
&= \langle X, X\rangle - \langle X, Y\rangle - \langle X, Y\rangle + \langle Y, Y\rangle \\
&= \|X\|^2 - 2\langle X, Y\rangle + \|Y\|^2
\end{aligned}$$

$\implies \|X\|^2 - 2\langle T(X), T(Y)\rangle + \|Y\|^2 = \|X\|^2 - 2\langle X, Y\rangle + \|Y\|^2$
$\implies -2\langle T(X), T(Y)\rangle = -2\langle X, Y\rangle$
$\implies \langle T(X), T(Y)\rangle = \langle X, Y\rangle \cdots\cdots\cdots\cdots (I)$
T.P.T T is linear transformation
Let $\{e_1, e_2, \ldots, e_n\}$ be a orthonormal basis of V.

$$\begin{aligned} \langle T(e_i), T(e_j)\rangle = \langle e_i, e_j\rangle &= 1, \qquad \text{if } i = j \\ &= 0 \qquad \text{if } i \neq j \end{aligned}$$

Therefore, $\{T(e_1), T(e_2), \ldots, T(e_n)\}$ be a orthonormal basis of V. Consider any $x \in V \implies x = a_1e_1 + a_2e_2 + \cdots + a_ne_n$, for some $a_i \in \mathbb{R}$
Consider $\langle x, e_i\rangle$
$= \langle a_1e_1 + a_2e_2 + \cdots + a_ne_n, e_i\rangle$
$= \langle a_1e_1, e_i\rangle + \langle a_2e_1, e_i\rangle + \cdots + \langle a_ie_i, e_i\rangle + \cdots + \langle a_ne_n, e_i\rangle$
$= a_1\langle e_1, e_i\rangle + a_2\langle e_1, e_i\rangle + \cdots + a_i\langle e_i, e_i\rangle + \cdots + a_n\langle e_n, e_i\rangle$
$= a_1(0) + a_2(0) + \cdots + a_i(1) + \cdot + a_n(0)$
$= a_i$
$a_i = \langle x, e_i\rangle,\ 1 \leq i \leq n$
$\implies x = \langle x, e_1\rangle e_1 + \langle x, e_2\rangle e_2 + \cdots + \langle x, e_n\rangle e_n$
Now, $x \in V \implies T(x) \in V$
$\implies T(x) = \langle T(x), T(e_1)\rangle T(e_1) + \langle T(x), T(e_2)\rangle T(e_2) + \cdots + \langle T(x), T(e_n)\rangle T(e_n)$
$= \langle x, e_1\rangle T(e_1) + \langle x, e_2\rangle T(e_2) + \cdots + \langle x, e_n\rangle T(e_n)$
$= a_1T(e_1) + a_2T(e_2) + \cdots + a_nT(e_n)$
$= \sum_{i=1}^n a_iT(e_i)$
$T(x) = \sum_{i=1}^n a_iT(e_i)$
Consider any $x, y \in V$ and $a, b \in \mathbb{R}$
T.P.T $T(ax + by) = aT(x) + bT(y)$
$x = \sum_{i=1}^n a_ie_i$, $y = \sum_{i=1}^n b_ie_i$ for some $a_i, b_j \in \mathbb{R}$.
$\implies T(x) = \sum_{i=1}^n a_iT(e_i),\ \ T(y) = \sum_{i=1}^n b_iT(e_i)$
Consider $T(ax + by)$
$= T(a\sum_{i=1}^n a_ie_i + b\sum_{i=1}^n b_ie_i)$
$= T(\sum_{i=1}^n aa_ie_i + \sum_{i=1}^n bb_ie_i)$
$= T(\sum_{i=1}^n (aa_i + bb_i)e_i)$
$= \sum_{i=1}^n (aa_i + bb_i)T(e_i)$
$= \sum_{i=1}^n aa_iT(e_i) + \sum_{i=1}^n bb_iT(e_i)$
$= a\sum_{i=1}^n a_iT(e_i) + b\sum_{i=1}^n b_iT(e_i)$
$= aT(x) + bT(y)$
$\implies T$ is linear transformation $\cdots\cdots\cdots\cdots (II)$
By (I) and $(II) \implies T$ is orthogonal linear transformation
we have, $T(x) = f(x) - x_0$
$\implies f(x) = T(x) + x_0 = L_{x_0}(T(x)) = (L_{x_0}OT)(x)$
$\implies f = L_{x_0}OT$, where T is O.L.T

Theorem 0.11 *Let V be an n dimensional inner product space and W be a subspace of V of dimension $n-1$. Let u be a unit vector orthogonal to W. Show that $T : V \to V$ defined* *by* $T(x) = x - 2\langle x, u\rangle u$ *is an orthogonal linear transformation such that* $T(w) = w; \forall w \in W$ *and* $T(u) = -u$.

Proof: Let V be an n dimensional inner product space and W be a subspace of V of dimension $n-1$. Let u be a unit vector orthogonal to W.
$T(x) = x - 2\langle x, u\rangle u$
T.P.T T is an orthogonal linear transformation
consider any $x, y \in V$ and $a, b \in \mathbb{R}$

$$\begin{aligned}
T(ax+by) &= ax + by - 2\langle ax+by, u\rangle u \\
&= ax + by - 2(\langle ax, u\rangle + \langle by, u\rangle)u \\
&= ax + by - 2(\langle ax, u\rangle u - 2\langle by, u\rangle)u \\
&= ax + by - 2a\langle x, u\rangle u - 2b\langle by, u\rangle u \\
&= ax - 2a\langle x, u\rangle u + by - 2b\langle y, u\rangle u \\
&= a(x - 2\langle x, u\rangle u) + b(y - 2\langle y, u\rangle u) \\
&= aT(x) + bT(y)
\end{aligned}$$

T.P.T $\langle T(x), T(y)\rangle = \langle x, y\rangle$
$\langle T(x), T(y)\rangle = \langle x - 2\langle x, u\rangle u, y - 2\langle y, u\rangle u\rangle$
$= \langle x, y\rangle - \langle x, 2\langle y, u\rangle u\rangle + \langle 2\langle x, u\rangle u, y\rangle + \langle 2\langle x, u\rangle u, 2\langle y, u\rangle u\rangle$
$= \langle x, y\rangle - 2\langle y, u\rangle\langle x, u\rangle - 2\langle x, u\rangle\langle u, y\rangle + 4\langle x, u\rangle\langle y, u\rangle\langle u, u\rangle$
$= \langle x, y\rangle - 2\langle x, u\rangle\langle y, u\rangle - 2\langle x, u\rangle\langle y, u\rangle + 4\langle x, u\rangle\langle y, u\rangle\langle u, u\rangle$
$= \langle x, y\rangle - 4\langle x, u\rangle\langle y, u\rangle + 4\langle x, u\rangle\langle y, u\rangle$, Since $\langle u, u\rangle = 1$
$= \langle x, y\rangle$
T is an orthogonal linear transformation
Now, $T(w) = w - 2\langle w, u\rangle u$
$\implies T(w) = w - (0)u$, since $\langle w, u\rangle = 0$
$\implies T(w) = w$
we have, $T(u) = u - 2\langle u, u\rangle u$
$\implies T(u) = u - 2(1)u$, since $\langle u, u\rangle = 1$
$\implies T(u) = u - 2u$
$\implies T(u) = -u$

Theorem 0.12 *State and prove the Cayley Hamilton Theorem.*

Statement: Let A be a $n \times n$ real matrix and let $f(x)$ be the characteristic polynomial of A then $f(A) = 0$. **Proof:** Let A be a $n \times n$ real matrix. let $f(x)$ be the characteristic polynomial of A
T.P.T $f(A) = 0$
characteristic polynomial of $A = \det(xI - A) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$
We have, $B \cdot \text{adj}B = \det B \cdot I$, where B is $n \times n$ matrix.
Let $B = xI - A$
So, $(xI - A)\text{adj}(xI - A) = \det(xI - A) \cdot I \cdots\cdots\cdots\cdots (i)$

Let adj$(B) = [D_{ij}]$, where $D_{ij} = (-1)^{i+j}\det(B_{ij})$. and B_{ij} is $n-1 \times n-1$ matrix obtained by deleting i^{th} row and j^{th} column of B.
So, D_{ij} = polynomial of degree $\leq n-1$.
Now,
adj$B$ = adj$(xI - A) = B_{n-1}x^{n-1} + B_{n-2}x^{n-2} + \cdots + B_0$, where B_k is $n \times n$ matrix.

$$(xI - A)adj(xI - A) = det(xI - A) \cdot I$$

$$(xI - A)(B_{n-1}x^{n-1} + B_{n-2}x^{n-2} + \cdots + B_0) = (x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0)I$$

$B_{n-1}x^n + B_{n-2}x^{n-2} + \cdots\cdots + B_0x - AB_{n-1}x^{n-1} - AB_{n-2}x^{n-2} - \cdots\cdots - AB_0 = x^nI + a_{n-1}x^{n-1}I + a_{n-2}x^{n-2}I + \cdots$

$\implies B_{n-1}x^n + (B_{n-2} - AB_{n-1})x^{n-1} + (B_{n-3} - AB_{n-2})x^{n-2} + \cdots + (B_0 - AB_1)x = x^nI + a_{n-1}x^{n-1}I + a_{n-2}x^{n-2}I + \cdots + a_0I$
$\implies B_{n-1} = I$
$B_{n-2} - AB_{n-1} = a_{n-1}I$
$B_{n-3} - AB_{n-2} = a_{n-2}I$
$\vdots$
$\vdots$
$B_0 - AB_1 = a_1I$
$-AB_0 = a_0I$
Multiplying $A^n, A^{n-1}, \ldots, A, I$ respectively to the equation from left, we get
$\implies A^nB_{n-1} = A^n$
$A^{n-1}B_{n-2} - A^nB_{n-1} = a_{n-1}A^{n-1}$
$A^{n-2}B_{n-3} - A^{n-1}B_{n-2} = a_{n-2}A^{n-2}$
$\vdots$
$\vdots$
$AB_0 - A^2B_1 = a_1A$
$-AB_0 = a_0I$
Adding above all equation, we get
$A^nB_{n-1} + A^{n-1}B_{n-2} - A^nB_{n-1} + AB_0 - A^2B_1 - AB_0 = A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1A + a_0I$
$\implies 0 = A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1A + a_0I$
$\implies f(A) = 0$