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TY1355 - Mathematics (P III) Topology of Metric Space

UNIT-I Metric spaces

UNIT-II Sequences and complete metric spaces

UNIT-III Compact sets

Question Bank

1. \mathbb{R} is open set
2. $(0, 1)$ open set
3. $[0, 1)$ not open
4. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ are not open (all sets)
5. $\mathbb{R} \setminus \mathbb{Q}$ not open set
6. $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ not open
7. $\{x\}, x \in \mathbb{R}$ not open
8. $[0, \alpha]$, $\alpha \leq 1$, open
9. Usual metric space $d(x, y) = |x - y|$
10. $X = \mathbb{Q}$, in \mathbb{R} , \mathbb{Q} is open
11. In discrete metric space, $\{x\}$ is open
12. In (X, d) discrete metric space, X, \emptyset are open
13. In (\mathbb{R}, d) , every open ball is an open set
14. In (X, d) , $d(x, y) = d(y, x) \forall x, y \in X$ called symmetry property
15. In (X, d) , $d(x, y) \geq 0$ for all $x, y \in X$, called non-negative property

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16. In (X, d)

$d(x, y) \leq d(x, z) + d(z, y) \forall x, y, z \in X$,
is called triangle inequality
property

17. A NLS is a vector space X with
a norm function defined on it.
If $\| \cdot \|$ is a norm function
defined on X , we say that
 $(X, \| \cdot \|)$ is a NLS.

18. Every real inner product space
is a NLS.

19. In (X, d) , bounded metric space is
 $d(x, y) \leq M \forall x, y \in X$; $M > 0$

20. In (X, d) ; open ball
 $B(p, r) = \{x \in X \mid d(x, p) < r\}$, $r > 0$

21. In (X, d) , closed ball $B[p, r]$
is denoted and defined as
 $B[p, r] = \{x \in X \mid d(x, p) \leq r\}$, $r > 0$

22. Every subset of discrete
metric space is open

23. In (X, d) , $G \subseteq X$, G is open if
 $\forall p \in G \exists r > 0$ s.t.
 $B(p, r) \subseteq G$ (open set G)

24. In (X, d) , every open ball in X
is an open set

25. In X , arbitrary union of open
set is open.

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26. In X , arbitrary union of open set is open

27. Convergent sequence

$d(x_n, x) < \epsilon$ for all $n > n_0$

$\Rightarrow (x_n) \subset X \rightarrow x$ if $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$
s.t.

$d(x_n, x) < \epsilon$ for all $n > n_0$

i.e. $(x_n) \rightarrow x$

28. Cauchy sequence In X , $(x_n) \in X$ is Cauchy if $\forall \epsilon > 0, \exists n_0 \in \mathbb{N}$ s.t.
 $d(x_m, x_n) < \epsilon$ for all $m, n > n_0$

29. Every Cauchy seq. in X is bounded

30. In X , $A \subseteq X$

$p \in \bar{A}$ iff $\exists (x_n) \in A \rightarrow p$

31. separability in metric space

$A \subseteq X$, A is countable iff \exists
an 1-1 function $f: A \rightarrow \mathbb{N}$

32. Separable In (X, d) , $A \subseteq X$

$\Rightarrow X$ is separable if X has a countable dense subset

33. A, B be dense subsets of (X, d) ,
and either A or B is open

$\Rightarrow A \cap B$ is dense in X .

34. Let (X, d) be a metric space with d as discrete metric then X is complete metric space.

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35. Closed subset of a complete metric space is a complete subspace.

36. Cantor's Intersection Theorem

Let (X, d) be a CMS, let $\{F_n\}_{n \in \mathbb{N}}$

be a sequence of non-empty closed subset of X s.t.

i) $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$

ii) $\text{diam}(F_n) \rightarrow 0$ as $n \rightarrow \infty$. Then $\bigcap_{n \in \mathbb{N}} F_n$ consists exactly one point.

37. Bolzano-Weierstrass Theorem

Every bounded real sequence has a convergent subsequence.

38. Intermediate Value Theorem:-

Let $f: [a, b] \rightarrow \mathbb{R}$ be a continuous real valued function.

Suppose $f(a) \cdot f(b) < 0$, then \exists

$p \in (a, b)$ s.t.

$$f(p) = 0$$

39. Every finite metric space is compact.

40. \mathbb{R} is not compact w.r.t. the usual metric.

41. Let $X = \mathbb{R}$, $d =$ usual metric, let

$$A = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\}$$

$\Rightarrow A$ is compact

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42. Let (X, d) be a m.s. Let $A, B \subseteq X$ be compact subsets of X then $A \cup B$ is compact
43. Any finite union of compact set is compact
44. Any compact subset of a metric space is closed and bounded
45. In (X, d) , $A, B \subseteq X$ be compact subsets of $X \Rightarrow A \cap B$ is compact
46. Every compact metric space has BWP
47. In (X, d) , X is sequentially compact if every sequence in X has a convergent subsequence
48. Every compact metric space is sequentially compact
49. Let $A, B \subseteq X$ s.t.
 A is closed and B is compact
 $\Rightarrow A \cap B$ is compact
50. Every sequentially compact metric space is compact
51. Heine-Borel Theorem
Every closed and bounded interval in \mathbb{R} is compact

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52. Let (X, d) be a m.s.; $(x_n) \in X$ converging to $p \in X$.

Let $A = \{x_n | n \in \mathbb{N}\} \cup \{p\}$.
 $\Rightarrow A$ is compact

53. Let $A \subseteq \mathbb{R} \Rightarrow A$ is sequentially compact iff A is closed & bounded.

54. A non-empty subset of \mathbb{R} is sequentially compact iff it has BWP

55. Let (X, d) be a m.s. & $A \subseteq X$ be compact & B is closed s.t. $A \cap B = \emptyset$ then $d(A, B) > 0$

56. $A \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded

57. Limit Point :- In X ,

$\forall r > 0, \exists x \neq p$ s.t.

$x \in B(p, r) \cap \{p\} \cap A \neq \emptyset$

$\Rightarrow p$ is limit point of $A \subseteq X$.

58. In (X, d) , $A \subseteq X$ then closure of A , denoted & defined as

$$\bar{A} = A \cup D(A)$$

$D(A) = \text{set of limit points}$
 $\text{in } X.$

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59. $\text{cl}(A)$ is the smallest closed set containing A

60. $\bar{A} = \bigcap \{F : F \text{ is closed, } F \supseteq A\}$

61. In (X, d) . $A, B \subseteq X$. Then

$$A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$$

$$\overline{A \cup B} = \bar{A} \cup \bar{B}$$

$$\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$$

$$\overline{\bar{A}} = A$$

65. Let (X, d) be a m.s. & let $A \subseteq X$ then boundary of A , is denoted and defined as
 $\partial(A) = b(A) = \bar{A} \cap \overline{X-A}$

66. closed set In (X, d) , let $F \subseteq X$. Then F is closed iff $X-F$ is open.

67. closed ball:
 $B[x_0, r] = \{x \in X \mid d(x, x_0) \leq r\}$

68. Every subset of discrete metric space is closed

69. In X , $A \subseteq X$, A is finite $\Rightarrow A$ is closed.

70. In X , $A \subseteq X$, A is closed iff $\bar{A} = A$

71. In X , $A \subseteq X$, A is closed iff A contains all its limit pts.

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72. Let (X, d) be a metric space. Let $A \subseteq X$. A point $p \in X$ is said to be an interior point of A if $\exists r > 0$ s.t.
$$B(p, r) \subseteq A$$

73. Let (X, d) be a m.s. Let $A, B \subseteq X$.
Then

74. $\underline{\Phi}^{\circ} = \underline{\Phi}$

75. $X^{\circ} = X$

76. If $A \subseteq B \Rightarrow A^{\circ} \subseteq B^{\circ}$

77. $(A \cap B)^{\circ} = A^{\circ} \cap B^{\circ}$

78. $A^{\circ} \cup B^{\circ} \subseteq (A \cup B)^{\circ}$

79. Hausdorff Properly

Let (X, d) be a m.s. Let $x, y \in X$
($x \neq y$), then \exists open sets U &
 V in X s.t.

$$x \in U, y \in V \text{ \& } U \cap V = \underline{\Phi}$$

80. Distance between two sets

Let $A, B \subseteq X$. Then

$$d(A, B) = \inf \{d(a, b) \mid a \in A, b \in B\}$$

81. In X , $A, B \subseteq X$ then A is open iff
$$A^{\circ} = A$$

82. Let (X, d) be a m.s. Let $A \subseteq X$ be
finite subset of X
 $\Rightarrow A$ is closed

83. In (X, d) , $\underline{\Phi}, X$ closed

84. Finite union of closed set is closed.

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- 85- Arbitrary intersection of closed set is closed.
- 86- Every finite subset of a m.s. does not have a limit pt
- 87- $\bar{A} = A \cup D(A)$, $A \subseteq X$.
- 88- closure of a set in metric space is a closed set
- 89- $A \subseteq X \Rightarrow D(A)$ is closed
- 90- In X , $A, B \subseteq X$. Then
 $A \subseteq B \Rightarrow \bar{A} \subseteq \bar{B}$
- 91- $\overline{A \cup B} = \bar{A} \cup \bar{B}$
- 92- $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$
- 93- Compact \Rightarrow closed & bounded
- 94- Every compact metric space has BWP.
- 95- every compact metric space is sequentially compact
- 96- sequentially compact m.s. is compact
- 97- discrete metric space
 $d(x, y) = 0$ if $x = y$
 $= 1$ if $x \neq y$
- 98- In X , $A \subseteq X \Rightarrow A^{\circ} \subseteq A$
- 99- $\text{ext}(A) = (A^{\circ})^{\circ}$
- 100- boundary of A is $D \cup X$, $A \subseteq X$.
 $D(A) = \text{fr}(A)$ but elements belong to the set itself.