

## Quotient Space and Orthogonal Transformations, Isometries, Cayley-Hamilton Theorem and its application

Q 1) Let  $V = \mathbb{R}^3$ ,  $W_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 0\}$  and  $W_2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 - x_2 + x_3 = 0\}$  are subspaces of  $V$ . then

- (a)  $\dim V/W_1 = \dim V/W_2 = 2, \dim W_2/W_1 \cap W_2 = 1$
- (b)  $\dim V/W_1 = \dim V/W_2 = 1, \dim W_2/W_1 \cap W_2 = 1$
- (c)  $\dim V/W_1 = \dim V/W_2 = 1, \dim W_2/W_1 \cap W_2 = 2$
- (d) None of the above.

Q 2) Let  $V = M_2(\mathbb{R})$ ,  $W_1 =$  Space of  $2 \times 2$  real symmetric matrices,  $W_2 =$  Space of  $2 \times 2$  real skew symmetric matrices.

- (a)  $\dim V/W_1 = 1, \dim V/W_2 = 1$
- (b)  $\dim V/W_1 = 2, \dim V/W_2 = 2$
- (c)  $\dim V/W_1 = 1, \dim V/W_2 = 3$
- (d) None of the above.

Q 3) Let  $V = P_2[x]$ , the space of polynomial of degree  $\leq 2$  over  $\mathbb{R}$  along with zero polynomial and  $W = \{f \in V : f(0) = 0\}$ . Then

- (a)  $\{\overline{1}, \overline{x+1}, \overline{(x+1)^2}\}$  is the basis of the quotient space  $V/W$ .
- (b)  $\{\overline{x+1}, \overline{x^2+1}\}$  is the basis of the quotient space  $V/W$
- (c)  $\{\overline{x+1}\}$  is the basis of the quotient space  $V/W$
- (d) None of the above.

Q 4) Let  $V$  be a real vector space and  $T : \mathbb{R}^6 \rightarrow V$  be a linear transformation such that  $S = \{Te_2, Te_4, Te_6\}$  spans  $V$ . Then, which of the following is true ?

- (a)  $S$  is a basis of  $V$
- (b)  $\{e_1 + \text{Ker}T, e_3 + \text{Ker}T, e_5 + \text{Ker}T\}$  is a basis of  $\mathbb{R}^6/\text{Ker}T$
- (c)  $\dim V/\text{Im}T \geq 3$
- (d)  $\dim \mathbb{R}^6/\text{Ker}T \leq 3$

Q 5) Consider  $W = \{(x, y, z) \in \mathbb{R}^3 : 2x+2y+z = 0, 3x+3y-2z = 0, x+y-3z = 0\}$ . Then  $\dim \mathbb{R}^3/W$  is

- (a) 1
- (b) 2
- (c) 3
- (d) 0

Q 6) Consider the linear transformation  $T : P_2[\mathbb{R}] \rightarrow M_2(\mathbb{R})$  defined by  $T(f) = \begin{pmatrix} f(0) - f(2) & 0 \\ 0 & f(1) \end{pmatrix}$  where  $P_2[\mathbb{R}] =$  space of polynomials of degree  $\leq 2$  along with 0 polynomial. Then

- (a)  $\ker T = 0$  and  $\dim(M_2(\mathbb{R})/\text{Im}T) = 3$
- (b)  $\dim(P_2[\mathbb{R}]/\text{Ker}T) = 1$
- (c)  $T$  is one-one and onto.
- (d)  $\dim(P_2[\mathbb{R}]/\text{Ker}T) = 2$

- Q 7) Let  $V = M_2(\mathbb{R})$  and  $W = \left\{ A \in M_2(\mathbb{R}) : A \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} A \right\}$ . Then
- (a)  $\dim V/W = 0$     (b)  $\dim V/W = 1$   
(c)  $\dim V/W = 2$     (d)  $\dim V/W = 3$
- Q 8) Let  $V = \mathbb{R}^4$  and  $W = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 = x_2 \text{ and } x_3 = x_4\}$  a subspace of  $V$ . Then
- (a)  $\{(\overline{1, 1, 0, 0}), \overline{(0, 1, 0, 1)}\}$  is the basis of  $V/W$ .  
(b)  $\{(\overline{1, 0, 1, 0}), \overline{(0, -1, 0, -1)}\}$  is the basis of  $V/W$   
(c)  $\{(\overline{1, 0, 1, 0}), \overline{(0, 1, 0, 1)}\}$  is the basis of  $V/W$   
(d) None of the above.
- Q 9) Let  $V = M_2(\mathbb{R})$ . Consider the subspaces  $W_1 = \left\{ \begin{pmatrix} a & -a \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R} \right\}$  and  $W_2 = \left\{ \begin{pmatrix} a & b \\ -a & d \end{pmatrix} : a, b, d \in \mathbb{R} \right\}$ . Then
- (a)  $\dim V/W_1 = \dim V/W_2 = 2, \dim W_2/W_1 \cap W_2 = 1$   
(b)  $\dim V/W_1 = \dim V/W_2 = 1, \dim W_2/W_1 \cap W_2 = 1$   
(c)  $\dim V/W_1 = \dim V/W_2 = 1, \dim W_2/W_1 \cap W_2 = 2$   
(d) None of the above.
- Q 10) Let  $V = M_2(\mathbb{R})$  and  $W = \{A \in M_2(\mathbb{R}) : \text{Tr}(A) = 0\}$  a subspace of  $V$ . Then
- (a)  $\left\{ \overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}, \overline{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \right\}$  is the basis of  $V/W$ .    (b)  $\left\{ \overline{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \right\}$  is the basis of  $V/W$   
(c)  $\left\{ \overline{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} \right\}$  is the basis of  $V/W$     (d) None of the above.
- Q 11) Let  $V = P_n[x]$ , the space of polynomials of degree  $\leq n$  over  $\mathbb{R}$  along with zero polynomial and  $D$  denote the linear transformation  $D : V \rightarrow P_{n-1}[x]$  defined by  $D(f) = \frac{df}{dx}$ . If  $W = \ker D$ , then
- (a)  $\dim V/W = n - 1$ .    (b)  $\dim V/W = 1$   
(c)  $\dim V/W = n$     (d) None of these.
- Q 12) Let  $A$  be a  $5 \times 7$  matrix over  $\mathbb{R}$ . Suppose  $\text{rank } A = 3$ . A linear transformation  $T : \mathbb{R}^7 \rightarrow \mathbb{R}^5$  is defined by  $T(X) = AX$ , where  $X$  is a column vector in  $\mathbb{R}^7$ , and  $W = \ker T$ ,  $U = \text{Im } T$ , then
- (a)  $\dim \mathbb{R}^7/W = 3, \dim \mathbb{R}^5/U = 2$ .    (b)  $\dim \mathbb{R}^7/W = 2, \dim \mathbb{R}^5/U = 2$ .  
(c)  $\dim \mathbb{R}^7/W = 2, \dim \mathbb{R}^5/U = 1$ .    (d) None of the above.
- Q 13) Let  $V = M_2(\mathbb{R})$  and  $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ . A linear transformation  $T : V \rightarrow V$  is defined by  $T(B) = AB - B$ . Then
- (a)  $T$  is a linear isomorphism.    (b)  $\dim V/\ker T = 1$ .  
(c)  $\dim V/\ker T = 2$ .    (d) None of these.

- Q 14) Let  $U, W$  be vector spaces over  $\mathbb{R}$  with bases  $\{u_1, u_2, \dots, u_m\}$  and  $\{w_1, w_2, \dots, w_n\}$  respectively. Let  $V = U \oplus W$  and linear transformation  $P_U : V \rightarrow U$  be defined by  $P_U(u + v) = u$ , where  $u \in U$  and  $w \in W$ . Then
- (a)  $\dim V / \ker P_U = n$ . (b)  $\dim V / \ker P_U = m$ .  
(c)  $\dim V / \ker P_U = m - n$ . (d) None of these.
- Q 15) Let  $V = \mathbb{R}^2, W = \{(x, y) \in \mathbb{R}^2 : y = x\}$ . Then
- (a)  $\{\overline{(1, 1)}\}$  is a bases of  $V/W$ . (b)  $\{\overline{(1, 0)}\}$  is a bases of  $V/W$ .  
(c)  $\{\overline{(1, 1)}, \overline{(1, -1)}\}$  is a bases of  $V/W$ . (d) None of the above.
- Q 16) If  $\alpha : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  and  $\beta : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  are translations such that  $\alpha((1, 1, 1, 1)) = (1, 0, -1, 3)$  and  $\beta((2, 2, 2, 2)) = (2, 0, 3, 4)$  then  $\alpha\beta(0, 0, 0, 0)$  is
- (a)  $(0, 0, 0, 0)$ . (b)  $(0, -3, -1, 4)$ . (c)  $(0, 3, 1, -4)$ . (d) None of these.
- Q 17) If  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an isometry defined by  $\alpha((x, y)) = (\frac{x}{2} + \frac{\sqrt{3}y}{2} - \frac{1}{2}, \frac{-\sqrt{3}x}{2} + \frac{y}{2} + \frac{\sqrt{3}}{2})$  and  $\alpha((x, y)) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$  then
- (a)  $x = 1, y = -1$ . (b)  $x = \sqrt{3}, y = 1$ . (c)  $x = 1, y = 1$ . (d) None of these.
- Q 18) Let  $\alpha$  be an orthogonal transformation of the plane such that the matrix of  $\alpha$  w. r. t. the standard basis of  $\mathbb{R}^2$  is  $\begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$ , then  $\alpha$  represents
- (a) a rotation about origin through  $\frac{\pi}{4}$ . (b) a rotation about origin through  $\frac{5\pi}{4}$ .  
(c) a rotation about the line  $y = -x$ . (d) None of the above.
- Q 19) Let  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  represents the rotation about origin by angle  $\frac{\pi}{4}$  and  $\beta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  represents a reflection about y-axis. Then  $\beta \circ \alpha$  represents
- (a) a rotation about origin through angle  $\frac{3\pi}{8}$ . (b) reflection in the line  $y = x$ .  
(c) a rotation about origin through angle  $\frac{\pi}{8}$ . (d) None of the above.
- Q 20) Let  $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  be an orthogonal transformation and  $E = \{v \in \mathbb{R}^3 : \alpha v = v\}$ . Then
- (a)  $\dim E = 1$  (b)  $\dim E \geq 1$   
(c) If  $\dim E = 2$ , then  $\alpha$  is reflection with respect to the plane.  
(d) None of the above.
- Q 21) Let  $\alpha : \mathbb{R}^3 \rightarrow \mathbb{R}^3$  represents reflection in the plane  $x + y + z = 0$ . The matrix of  $\alpha$  with respect to the standard basis of  $\mathbb{R}^3$  is
- (a)  $\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & -1 \end{pmatrix}$  (b)  $\frac{1}{3} \begin{pmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{pmatrix}$  (c)  $\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  (d) None of these.
- Q 22) Let  $V$  be an  $n$ -dimensional real inner product space. Suppose  $B = \{e_i\}_{i=1}^n$  and  $B' = \{f_i\}_{i=1}^n$  are orthogonal basis of  $V$ . Then
- (a) If  $T : V \rightarrow V$  is a linear transformation such that  $T(e_i) = f_i$  for  $i = 1$  to  $n$ , then  $T$  is orthogonal.

- (b) If  $T : V \rightarrow V$  is a linear transformation such that  $T(e_i) = f_i$  for  $i = 1$  to  $n$ , then  $T$  need not be orthogonal.
- (c) There exist a linear transformation  $T : V \rightarrow V$  such that  $\{T(e_i)\}_{i=1}^n$  is an orthogonal basis of  $V$ , but  $\{T(f_i)\}_{i=1}^n$  is not an orthonormal basis of  $V$ .
- (d) None of the above.

Q 23) Let  $A$  and  $B$  be  $n \times n$  real orthogonal matrices. Then

- (a)  $AB$  and  $A + B$  are orthogonal matrices.
- (b)  $AB$  and  $BA$  are orthogonal matrices.
- (c)  $A + B$  is an orthogonal matrix.
- (d) None of the above.

Q 24) Let  $A, B$  be  $n \times n$  real matrices. If  $A$  and  $AB$  are orthogonal matrices, then

- (a)  $B$  is orthogonal but  $BA$  may not be orthogonal
- (b)  $B$  and  $BA$  both are orthogonal matrices.
- (c)  $B$  may not be orthogonal matrix.
- (d) None of the above.

Q 25) Let  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be an isometry fixing origin and  $\alpha \neq$  identity. Then

- (a)  $\alpha((1, 0))$  is in the first quadrant.
- (b)  $\alpha((1, 0)) \in \{(-1, 0), (0, 1), (0, -1)\}$ .
- (c)  $\alpha((1, 0))$  lies on the unit circle  $S^1$ .
- (d) None of the above.

Q 26) If  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear transformation such that  $\langle v, w \rangle = 0 \Rightarrow \langle \alpha(v), \alpha(w) \rangle = 0$   $\forall v, w \in \mathbb{R}^2$ . Then

- (a)  $\alpha$  is an isometry of  $\mathbb{R}^2$ .
- (b)  $\alpha$  is an orthogonal transformation.
- (c)  $\alpha = aT$  where  $T$  is an orthogonal transformation and  $a \in \mathbb{R}$ .
- (d) None of the above.

Q 27) Let  $\alpha : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  be defined by  $\alpha((x, y)) = (ax + by + e, cx + dy + f)$  where  $a, b, c, d, e, f \in \mathbb{R}$ . Then  $\alpha$  is an isometry if and only if

- (a)  $ad - bc \neq 0, e, f > 0$
- (b)  $ad - bc = \pm 1$ .
- (c)  $a^2 + c^2 = 1, b^2 + d^2 = 1, ab + cd = 0$ .
- (d) None of the above.

Q 28) Let  $V$  be a finite dimensional inner product space and  $\alpha : V \rightarrow V$  be an isometry. Then

- (a)  $\alpha$  is one-one may not be onto.
- (b)  $\alpha$  is one-one only if  $\alpha(0) = 0$ .
- (c)  $\alpha$  is bijective.
- (d) None of the above.

Q 29) Let  $A = \begin{pmatrix} 10 & -9 \\ 4 & -2 \end{pmatrix}$ , then

- (a)  $A^{-1} = \frac{1}{16}[A + 8I]$  (b)  $A^{-1} = \frac{1}{16}[A - 8I]$   
(c)  $A^{-1} = \frac{1}{16}[-A + 8I]$  (d)  $A^{-1} = \frac{1}{16}[-A - 8I]$

Q 30) The following pairs of  $n \times n$  matrices do not have same characteristic polynomial.

- (a)  $A$  and  $A^t$ . (b)  $A$  and  $PAP^{-1}$  where  $P$  is non singular  $n \times n$  matrix.  
(c)  $A$  and  $A^2$ . (d)  $AB$  and  $BA$ .

Q 31) Let  $p(t) = t^2 + bt + c$  where  $b, c \in \mathbb{R}$ . Then the number of real matrices having  $p(t)$  as characteristic polynomial is

- (a) One (b) Two  
(c) Infinity (d) None of the above

Q 32) Let  $p(t) = t^3 - 2t^2 + 5$  be the characteristic polynomial of  $A$  then  $\det A$  and  $\text{tr} A$  are

- (a) 5, -2 (b) 2, 5  
(c) -5, 2 (d) -2, 5

Q 33) If  $A$  is a  $3 \times 2$  matrix over  $\mathbb{R}$  and  $B$  is a  $2 \times 3$  matrix over  $\mathbb{R}$  and  $p(t)$  is the characteristic polynomial of  $AB$ , then

- (a)  $t^3$  divides  $p(t)$  (b)  $t^2$  divides  $p(t)$   
(c)  $t$  divides  $p(t)$  (d) None of the above

Q 34) Let  $A$  and  $B$  be  $n \times n$  matrix over  $\mathbb{R}$  such that  $\text{tr} A = \text{tr} B$  and  $\det A = \det B$ . Then

- (a) Characteristic polynomial of  $A$  = Characteristic polynomial of  $B$ .  
(b) Characteristic polynomial of  $A \neq$  Characteristic polynomial of  $B$ .  
(c) Characteristic polynomial of  $A$  = Characteristic polynomial of  $B$  if  $n = 3$ .  
(d) Characteristic polynomial of  $A$  = Characteristic polynomial of  $B$  if  $n = 2$ .

Q 35) Let  $A$  and  $B$  be  $n \times n$  matrix over  $\mathbb{R}$  such that characteristic polynomial of  $A$  = characteristic polynomial of  $B$ . Then

- (a)  $A$  and  $B$  are similar matrices (b)  $\det A = \det B$   
(c)  $AB = BA$  (d) None of the above.

Q 36) Let  $p(t) = t^3 - 2t^2 + 15$  be the characteristic polynomial of  $A$ . Then  $\det A$

- (a) 15 (b) -15 (c) 0 (d) None of these

Q 37) Let  $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$  (a)  $A^{10} = \begin{pmatrix} 2^{10} & -2^{10} \\ -2^{10} & 2^{10} \end{pmatrix}$  (b)  $A^{10} = \begin{pmatrix} 2^{11} & -2^{11} \\ -2^{11} & 2^{11} \end{pmatrix}$   
(c)  $A^{10} = \begin{pmatrix} 2^9 & -2^9 \\ -2^9 & 2^9 \end{pmatrix}$  (d)  $A^{10} = \begin{pmatrix} -2^9 & 2^9 \\ 2^9 & -2^9 \end{pmatrix}$

Q 38) Let  $A$  be a  $3 \times 3$  matrix and  $\lambda_1, \lambda_2$  be only two distinct eigen values of  $A$ . Then its characteristic polynomial  $k_A(x)$  is

- (a)  $(x - \lambda_1)(x - \lambda_2)$

- (b)  $(x - \lambda_1)(x - \lambda_2)^2$   
 (c)  $(x - \lambda_1)^2(x - \lambda_2)$   
 (d)  $(x - \lambda_1)^2(x - \lambda_2)$  or  $(x - \lambda_1)(x - \lambda_2)^2$

Q 39) Let characteristic polynomial of  $A$  is  $t^2 + a_1t + a_0$  and characteristic polynomial of  $A^{-1}$  is  $t^2 + a'_1t + a'_0$ . Then

- (a)  $a_0a'_0 = 1$  and  $a_1 + a'_1 = 1$  (b)  $a_1a'_1 = 1$  and  $a_0a'_0 = 1$   
 (c)  $a_0a'_0 = 1$  (d)  $a_0a'_0 = 1$  and  $a'_1 = a_1a'_0$

Q 40) If  $p_1(t) = t^2 + a_1t + a_0$  is characteristic polynomial of  $A$  and  $p_2(t) = t^2 + a'_1t + a'_0$  is characteristic polynomial of  $A^2$  then

- (a)  $a'_1 = a_1^2$  and  $a'_0 = a_0^2$  (b)  $a'_1 = 2a_1$  and  $a'_0 = a_0^2$   
 (c)  $a'_0 = a_0^2$ ,  $a'_1 = a_1^2 - 2a_0$  (d) None of the above

Q 41) Let  $A_{6 \times 6}$  be a matrix with characteristic polynomial  $x^2(x - 1)(x + 1)^3$ , then trace  $A$  and determinant of  $A$  are

- (a) -2, 0 (b) 2, 0 (c) 3, 1 (d) 3, 0

Q 42)  $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$  and  $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$  are similar (non-zero  $a, b, d$ )

- (a) for any reals  $a, b, d$ . (b) if  $a = d$ .  
 (c) if  $a \neq d$ . (d) never similar.

Q 43) Let  $A_{6 \times 6}$  be a diagonal matrix over  $\mathbb{R}$  with characteristic polynomial  $(x - 2)^4(x + 3)^2$ . Let  $V = \{B \in M_6(\mathbb{R}) : AB = BA\}$ . Then  $\dim V =$

- (a) 8 (b) 12 (c) 6 (d) 20.

Q 44) If  $A - I_n$  is a  $n \times n$  nilpotent matrix over  $\mathbb{R}$ , then characteristic polynomial of  $A$  is

- (a)  $(t - 1)^n$  (b)  $t^n$   
 (c)  $t^n - 1$  (d)  $(t^{n-1} - 1)t$

Q 45) If  $A \in M_2(\mathbb{R})$ ,  $\text{tr } A = -1$ ,  $\det A = -6$  then  $\det(I_2 + A)$  is

- (a) -6 (b) -5 (c) -1 (d) None of the above.

Q 46) Let  $A = [a_{ij}]_{10 \times 10}$  be a real matrix such that  $a_{i,i+1} = 1$  for  $1 \leq i \leq 9$  and  $a_{ij} = 0$  otherwise, then

- (a)  $A^9(A - I)$  (b)  $(A - I)^{10}$   $A^{10} = 0$   $A(A - I)^9 = 0$

Q 47)  $T : \mathbb{R}^4 \rightarrow \mathbb{R}^4$  is a linear transformation such that  $T^3 + 3T^2 = 4I$ . If  $S = T^4 + 3T^3 - 4I$ , then

- (a)  $S$  is not one-one. (b)  $S$  is one-one.  
 (c) if 1 is not an eigen value of  $T$  then  $S$  is invertible.  
 (d) None of these.

Q 48) Which of the following statements are true

1. If the characteristic roots of two  $n \times n$  matrices are same then their characteristic polynomials are same.
2. If the characteristic polynomials of two  $n \times n$  matrices are same then their characteristic roots are same.
3. If eigen values of two  $n \times n$  matrices are same then their eigen vectors are same.
4. The characteristic roots of two  $n \times n$  matrices are same but their characteristic polynomials may not be same.

- (a) ii and iv are true.    (b) i, iii are true.  
(c) i, ii and iii are true.    (d) only ii is true.

Q 49) A  $2 \times 2$  matrix  $A$  has the characteristic polynomial  $x^2 + 2x - 1$ , then the value of  $\det (2I_2 + A)$  is

- (a)  $\frac{1}{\det A}$     (b) 0  
(c)  $2 + \det A$     (d)  $2 \det A$

Q 50) If  $A$  and  $B$  are  $n \times n$  then trace of  $I - AB + BA$  is

- (a) 0    (b)  $n$     (c)  $2 \operatorname{tr} AB$     (d) None of these.

## Diagonalization of a matrix and Orthogonal Diagonalization and Quadratic Form

1. If  $A$  and  $B$  are  $3 \times 3$  matrices over  $\mathbb{R}$  having  $(1, -1, 0)^t$ ,  $(1, 1, 0)^t$ , and  $(0, 0, 1)^t$  as eigenvectors. Then
  - (a)  $A$  and  $B$  are similar matrices.
  - (b)  $AB = BA$ .
  - (c)  $A$  and  $B$  have same eigenvalues.
  - (d) None of the above.
2. Let  $A = \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix}$ . Then,
  - (a)  $A$  and  $A^{100}$  are both diagonalizable.
  - (b)  $A$  is diagonalizable but  $A^{100}$  is not.
  - (c) Neither  $A$  nor  $A^{100}$  is diagonalizable.
  - (d) None of the above.
3. Let  $A = \begin{pmatrix} 1 & 2 & 4 \\ 0 & -1 & -2 \\ 0 & 0 & 3 \end{pmatrix}$  and  $B = A^{100} + A^{20} + I$ . Then,
  - (a)  $A, B$  are not diagonalizable.
  - (b)  $A$  is diagonalizable, but  $B$  is not diagonalizable.
  - (c)  $AB$  is diagonalizable
  - (d) None of the above.
4. If  $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear transformation such that  $T(61, 23) = (189, 93)$  and  $T(67, 47) = (195, 117)$ . Then
  - (a)  $T$  is diagonalizable with distinct eigenvalues.
  - (b)  $T$  is not diagonalizable.
  - (c)  $T$  does not have distinct eigenvalues, but is diagonalizable.
  - (d) None of the above.
5. Which of the following matrices is not diagonalizable?
  - (a)  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$
  - (b)  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  - (c)  $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$
  - (d)  $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$
6. Let  $A$  be a  $n \times n$  real orthogonal matrix. Then
  - (a)  $A$  has  $n$  real eigen values and each eigen value is  $\pm 1$ .
  - (b)  $A$  is diagonalizable
  - (c)  $A$  may not have any real eigen value.
  - (d)  $A^2 = I$
7. Let  $A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$ , then  $A$  is diagonalizable if
  - (a)  $a = b, c = 1$
  - (b)  $a = 1 = b = c$
  - (c)  $a = b = c = 0$
  - (d)  $a, b, c > 0$
8. Let  $A = \begin{bmatrix} 0 & a \\ 0 & -a \end{bmatrix}$ 
  - (a)  $A$  is diagonalizable but not orthogonally diagonalizable.
  - (b)  $A$  is not diagonalizable for any  $a \in \mathbb{R}$ .
  - (c)  $A$  is orthogonally diagonalizable if and only if  $a = 1$
  - (d) None of these.
9. If  $A$  is a  $4 \times 4$  matrix having all diagonal entries 0, then
  - (a) 0 is an eigenvalue of  $A$ .
  - (b)  $A^4 = 0$
  - (c)  $A$  is not diagonalizable.
  - (d) None of these.
10. Let  $A$  be an  $n \times n$  non-zero nilpotent matrix over  $\mathbb{R}$ . Then
  - (a)  $A$  is diagonalizable.
  - (b)  $A$  is diagonalizable if  $n$  is odd.
  - (c)  $A$  is not diagonalizable.
  - (d) None of the above.

11. Let  $A = \begin{pmatrix} \alpha & -3 \\ 3 & 0 \end{pmatrix}$ ,  $\alpha \in \mathbb{R}$  is a parameter. Then  
 (a)  $A$  is not diagonalizable for any  $\alpha \in \mathbb{R}$ . (b)  $A$  is diagonalizable  $\forall \alpha \in \mathbb{R}$ .  
 (c)  $A$  is not diagonalizable if  $-6 \leq \alpha \leq 6$ . (d)  $A$  is diagonalizable if  $-6 < \alpha < 6$ .
12. Let  $A$  and  $B$  be  $n \times n$  matrices over  $\mathbb{R}$  such that  $AB = A - B$ . If  $B$  is a diagonalizable matrix with only one eigenvalue 2, then,  
 (a) 2 is also an eigenvalue of  $A$ . (b)  $A$  is diagonalizable and  $-2$  is the only eigenvalue of  $A$ .  
 (c)  $A$  may not be diagonalizable. (d) None of these.
13. The matrix  $A = \begin{pmatrix} 1 & 7 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 2 \end{pmatrix}$   
 (a) Not diagonalizable. (b) is similar to  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}$   
 (c) is similar to  $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ . (d) None of the above.
14. Let  $A, B, C$  be  $3 \times 3$  non-diagonal matrices over  $\mathbb{R}$  such that  $A^2 = A, B^2 = -I, (C - 3I)^2 = 0$ . Then  
 (a)  $A, B, C$  are all diagonalizable over  $\mathbb{R}$ . (b)  $A, C$  are all diagonalizable over  $\mathbb{R}$ .  
 (c) Only  $A$  is diagonalizable over  $\mathbb{R}$ . (d) None of the above
15. Let  $A \in M_3(\mathbb{R})$  such that  $AB = BA$  for all  $B \in M_3(\mathbb{R})$ . Then  
 (a)  $A$  has distinct eigenvalues and is diagonalizable.  
 (b)  $A$  is not diagonalizable.  
 (c)  $A$  does not have distinct eigenvalues but is diagonalizable.  
 (d) None of the above.
16. If  $A, B, C, D \in M_2(\mathbb{R})$  such that  $A, B, C, D$  are non-zero and not diagonal. If  $A^2 = I, B^2 = B, C^2 = 0, C \neq 0$  and every eigenvalue of  $D$  is 2, then  
 (a)  $A, B, C, D$  are all diagonalizable. (b)  $B, C, D$  are diagonalizable.  
 (c)  $A, B$  are diagonalizable. (d) Only  $D$  is diagonalizable.
17. If  $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$  and  $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$  then  
 (a) Both  $A, B$  are diagonalizable,  $A$  is also orthogonally diagonalizable.  
 (b) Both  $A, B$  are orthogonally diagonalizable.  
 (c) Both  $A, B$  are diagonalizable,  $B$  is also orthogonally diagonalizable.  
 (d) Both  $A, B$  are diagonalizable, but both  $A, B$  are not orthogonally diagonalizable.

18. If  $v = [1, 0, 1]$  is a row vector then,  
 (a)  $v^t v$  is not orthogonally diagonalizable.  
 (b)  $vv^t v$  is orthogonally diagonalizable.  
 (c)  $v^t v$  is not diagonalizable.  
 (d) None of the above.
19. Let  $A$  be an  $m \times n$  matrix over  $\mathbb{R}$ . Then  
 (a)  $AA^t$  is not orthogonally diagonalizable.  
 (b)  $I_m + AA^t$  is not orthogonally diagonalizable.  
 (c)  $AA^t$  and  $A^t A$  are orthogonally diagonalizable. (d) None of the above.
20. Let  $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ . If  $P^t A P = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ , then  $P =$   
 (a)  $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$  (b)  $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$  (c)  $\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$  (d) None of the above.
21. Let  $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ ,  $a \in \mathbb{R}$ . Then  
 (a)  $A$  is not diagonalizable for any  $a \in \mathbb{R}$ .  
 (b)  $A$  is diagonalizable but not orthogonally diagonalizable.  
 (c)  $A$  is orthogonally diagonalizable if and only if  $a = 0$ . (d) None of the above.
22. The equation  $2x^2 - 4xy - y^2 - 4x + 10y - 13 = 0$  after rotation and translation can be reduced to  
 (a) an ellipse (b) a hyperbola (c) a parabola (d) a pair of straight lines.
23. The conic  $x^2 + 2xy + y^2 = 1$  reduces to the standard form after rotation through a angle  
 (a)  $\frac{\pi}{4}$  (b)  $\frac{\pi}{3}$  (c)  $\frac{2\pi}{3}$  (d)  $\frac{\pi}{6}$
24. The quadratic form  $Q(x) = x_1^2 + 4x_1x_2 + x_2^2$  has  
 (a) rank = 1, signature = 1. (b) rank = 2, signature = 0.  
 (c) rank = 2, signature = 2. (d) None of the above.
25. Let  $A$  be a  $4 \times 4$  real symmetric matrix. Then there exists a  $4 \times 4$  real symmetric matrix  $B$  such that  
 (a)  $B^2 = A$  (b)  $B^3 = A$  (c)  $B^4 = A$  (d) None of these
26. The matrix  $\begin{pmatrix} 1 & 2 \\ 2 & k \end{pmatrix}$  is positive definite if  
 (a)  $k > 4$  (b)  $-2 < k < 2$  (c)  $|k| > 2$  (d) None of these.
27.  $ax^2 + bxy + cy^2 = d$  where  $a, b, c$  are not all zero and  $d > 0$  represents  
 (a) ellipse if  $b^2 - 4ac > 0$  and hyperbola if  $b^2 - 4ac < 0$ .  
 (b) ellipse if  $b^2 - 4ac < 0$  and hyperbola if  $b^2 - 4ac > 0$ .  
 (c) is a circle if  $b = 0$  and  $a = c$  else it is a hyperbola.  
 (d) None of these.

28. The conic  $x^2 + 10x + 7y = -32$  represents  
 (a) a hyperbola (b) an ellipse. (c) a parabola (d) a pair of straight lines.
29. For the quadratic form  $Q(x) = 2x_1^2 + 2x_2^2 - 2x_1x_2$   
 (a) rank = 2, signature = 1 (b) rank = 1, signature = 1  
 (c) rank = 2, signature = 0 (d) rank = 2, signature = 2
30. For the quadratic form  $Q(x) = -3x_1^2 + 5x_2^2 + 2x_1x_2$ ,  
 (a) rank = 2, signature = 0 (b) rank = 2, signature = 1  
 (c) rank = 2, signature = 2 (d) rank = 1, signature = 1
31. The symmetric matrix associated to the quadratic form  $5(x_1 - x_2)^2$  is,  
 (a) positive definite (b) positive semi definite (c) indefinite (d) negative definite.
32. The quadratic form  $Q(x) = 2x_1^2 - 4x_1x_2 - x_2^2$  after rotation can be reduced to standard form  
 (a)  $3y_1^2 - 2y_2^2$  or  $2y_1^2 + 3y_2^2$  (b)  $3y_1^2 + 2y_2^2$  (c)  $-3y_1^2 + 2y_2^2$  (d)  $2y_1^2 - 4y_2^2$
33. The equation  $x^2 + y^2 + z^2 - 2x + 4y - 6z = 11$  represents  
 (a) None of the below (b) a hyperboloid of one sheet  
 (c) a hyperboloid of two sheet (d) a sphere.
34. The conic  $3x^2 - 4xy = 2$  represents  
 (a) an ellipse (b) a hyperbola (c) a parabola (d) a pair of straight lines.
35. Let  $Q(X) = X^tAX$ , where  $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ ,  $X = (x_1, x_2, x_3, x_4)^t$ . Then by orthogonal change of variable,  $Q(X)$  can be reduced to  
 (a)  $y_1y_2 + y_3^2$  (b)  $y_1y_2 + y_2^2 + y_3^2$   
 (c)  $y_1^2 + y_2^2 + y_3^2 - y_4^2$  (d)  $y_2^2 + y_2^2 - y_3y_4$
36. If  $A_{n \times n}$  be real matrix then which of the following is true-  
 (a)  $A$  has at least one eigen value. (b)  $\forall X, Y \in \mathbb{R}, \langle AX, AY \rangle > 0$   
 (c) Each eigen value of  $A^tA \geq 0$  (d)  $A^tA$  has  $n$  eigen values.

# Groups, Subgroups, Lagrange's Theorem, Cyclic Groups and Groups of Symmetry

- (1) Let  $G$  be a group and  $a_1, a_2, a_3, a_4, a_5 \in G$ . Then the inverse of  $a_1 a_2^{-1} a_3 a_4^{-1} a_5$  is
- (a)  $a_1^{-1} a_2 a_3^{-1} a_4 a_5^{-1}$       (b)  $a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} a_5^{-1}$       (c)  $a_5^{-1} a_4 a_3^{-1} a_2 a_1^{-1}$       (d)  $a_5^{-1} a_4^{-1} a_3^{-1} a_2^{-1} a_1^{-1}$
- (2) Let  $G$  be a group and  $a, b, c \in G$ . Consider the equations  $axb = c$  and  $a^{-1}y^{-1}b^{-1} = c$ . The equations have solutions:
- (a)  $x = a^{-1}cb^{-1}, y = acb$       (b)  $x = acb, y = a^{-1}c^{-1}b^{-1}$   
 (c)  $x = a^{-1}cb^{-1}, y = b^{-1}c^{-1}a^{-1}$       (d) None of the above.
- (3) Let  $O$  denote the set of odd integers. Then
- (a)  $O$  forms a group under the operation of addition  
 (b)  $O$  forms a group under the operation of multiplication  
 (c)  $O$  does not form a group under addition as  $O$  is not closed under addition.  
 (d) None of the above.
- (4) Consider the set  $G = \{\bar{5}, \bar{15}, \bar{25}, \bar{35}\} \pmod{40}$  under multiplication of residue classes modulo 40. Then
- (a)  $G$  is not a group as  $\bar{1} \notin G$ .      (b)  $G$  is a group with  $\bar{25}$  as identity element.  
 (c)  $G$  is not a group as  $\bar{5}$  has no inverse in  $G$ .      (d) None of the above.
- (5) Consider the equation  $ax = b$  in the group  $S_3$ , where  $a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$  and  $b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$  (the operation of composite of maps being from left to right). Then
- (a)  $x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$       (b)  $x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$       (c)  $x = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$       (d) None of these.
- (6) Consider the group  $(\mathbb{Z}, \circ)$ , where  $\mathbb{Z}$  is the set of integers and  $a \circ b = a + b - 5$ .
- (a)  $-5$  is the identity element of  $(\mathbb{Z}, \circ)$  and the inverse of  $a \in \mathbb{Z}$  is  $5 - a$ .  
 (b)  $5$  is the identity element of  $(\mathbb{Z}, \circ)$  and the inverse of  $a \in \mathbb{Z}$  is  $10 - a$ .  
 (c)  $5$  is the identity element of  $(\mathbb{Z}, \circ)$  and the inverse of  $a \in \mathbb{Z}$  is  $25 - a$ .  
 (d) None of the above.
- (7) Consider the group  $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{Q}, a \neq 0 \right\}$  under multiplication of  $2 \times 2$  matrices. Then the identity element of the group  $G$  is
- (a)  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$       (b)  $\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$       (c)  $\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$       (d)  $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
- (8) Let  $X$  be a non-empty set and  $\mathcal{P}(X)$  denote the power set of  $X$ . For the group  $(\mathcal{P}(X), \Delta)$  ( $\Delta$  being the symmetric difference)

- (a)  $X$  is the identity element and for  $A \in \mathcal{P}(X)$ ,  $A^{-1} = A$ .  
 (b)  $\emptyset$  is the identity element and for  $A \in \mathcal{P}(X)$ ,  $A^{-1} = A^c$ .  
 (c)  $\emptyset$  is the identity element and for  $A \in \mathcal{P}(X)$ ,  $A^{-1} = A$ .  
 (d) None of the above.
- (9) Let  $G = GL_n(\mathbb{R})$ . Then  
 (a)  $G$  is an infinite abelian group for  $n = 2$ .  
 (b)  $G$  is an infinite non-abelian group for  $n \geq 2$ .  
 (c)  $G$  is an infinite group with identity element  $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ .  
 (d) None of the above.
- (10) Suppose  $H$  is a proper subgroup of  $\mathbb{Z}$  under addition and  $12, 14$  and  $18 \in H$ , then  
 (a)  $H = 756\mathbb{Z}$                       (b)  $H = 2\mathbb{Z}$                       (c)  $H = 4\mathbb{Z}$                       (d)  $H = 44\mathbb{Z}$
- (11) Let  $G$  be a group having exactly 8 elements of order 3. Then  
 (a)  $G$  has exactly 8 subgroups of order 3.                      (b)  $G$  has exactly 4 subgroups of order 3.  
 (c)  $G$  has exactly 6 subgroups of order 3.                      (d) None of the above.
- (12) Let  $G$  be an abelian group and  $G$  has an element of order 4 and an element of order 5. Then  
 (a)  $G$  has a subgroup of order 20 but may not have a subgroup of order 10.  
 (b)  $G$  has a subgroup of order 10.  
 (c)  $G$  does not have a subgroup of order 20.  
 (d) None of the above.
- (13) Let  $G$  be a cyclic group of order 4000. Then  $G$  has  
 (a) 400 elements of order 10.                      (b) 40 elements of order 10.  
 (c) 4 elements of order 10.                      (d) None of the above.
- (14) Let  $G$  be a group and  $a \in G$ . If  $o(a) = 30$ . Then, the number of distinct right cosets of  $\langle a^4 \rangle$  in  $\langle a \rangle$  is  
 (a) 6                      (b) 2                      (c) 15                      (d) None of these.
- (15) Let  $G = S_3$  and consider the subgroup  $H = \{I, (12)\}$  of  $G$ . Then  
 (a) Every left coset of  $H$  in  $G$  is also a right coset.  
 (b)  $[G : H] = 3$  and two left cosets of  $H$  in  $G$  are also right cosets.  
 (c) No left coset except  $H$  itself is a right coset of  $H$  in  $G$ .  
 (d) None of the above.
- (16) Let  $G = \mathbb{Z}$  and  $H = 5\mathbb{Z}$ . Then the following pair of left cosets are not equal. Then  
 (a)  $11 + 5\mathbb{Z}$  and  $-39 + 5\mathbb{Z}$ .                      (b)  $11 + 5\mathbb{Z}$  and  $-25 + 5\mathbb{Z}$ .  
 (c)  $11 + 5\mathbb{Z}$  and  $-34 + 5\mathbb{Z}$ .                      (d) None of these.
- (17) Let  $G$  be a group and  $a, b \in G$  such that  $ab = ba$ ,  $o(a) = m$ ,  $o(b) = n$ . Then

- (a)  $o(ab) = mn$   
 (b)  $o(ab) = \text{l.c.m.}[m, n]$   
 (c)  $o(ab)$  divides  $\text{l.c.m.}[m, n]$  but may not be equal to  $\text{l.c.m.}[m, n]$   
 (d) None of the above.
- (18) Consider the Quaternion group  $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ , where  $ij = -ji = k$ ,  $jk = -kj = i$ ,  $ki = -ik = j$  and  $i^2 = j^2 = k^2 = -1$ . Then
- (a)  $Q$  has exactly 3 subgroups of order 2.      (b)  $Q$  has exactly 2 subgroups of order 4.  
 (c)  $Q$  has exactly 1 subgroup of order 2.      (d) None of the above.
- (19) Let  $G$  be a cyclic group of order  $n$  generated by  $a$ . For non-zero integers  $r, s$  we have  $\langle a^r \rangle \subseteq \langle a^s \rangle$  if and only if
- (a)  $r \mid s$       (b)  $\gcd(n, s) \mid \gcd(n, r)$       (c)  $s \mid r$       (d) None of these.
- (20) The group  $10\mathbb{Z} \cap 15\mathbb{Z}$  is generated by
- (a) 5 and  $-5$       (b) 60 and  $-60$       (c) 30 and  $-30$       (d) None of these.
- (21) Let  $G$  be a group and  $a, b \in G$ . If  $o(a) = 18$  and  $o(b) = 12$ . Then  $\langle a \rangle \cap \langle b \rangle$  has order
- (a) 1      (b) 1 or 6      (c) 6      (d) 36.
- (22) Let  $H$  be the smallest subgroup of  $(\mathbb{Z}_{40}, +)$  containing  $\overline{24}$  and  $\overline{10}$ . Then  $H$  is generated by
- (a) 36      (b) 37      (c) 38      (d) 39.
- (23) If  $G$  is a group having exactly one proper non-trivial subgroup, then order of  $G$  is
- (a) even      (b)  $p^2$  where  $p$  is a prime      (c) odd      (d)  $pq$  where  $p, q$  are distinct primes.
- (24) Let  $G$  be an abelian group of order 10 and  $S = \{g \in G : g \neq g^{-1}\}$ . Then  $S$  has
- (a) 2 elements      (b) 4 elements      (c) 8 elements      (d) None of these
- (25) The group of symmetries of a square has order
- (a) 4      (b) 24      (c) 8      (d) None of these
- (26) The group of symmetries of
- (a) a square is abelian.      (b) a rectangle is abelian.  
 (c) an equilateral triangle is abelian.      (d) None of the above.
- (27) Let  $G$  be the group of symmetries of a square. The center of  $G$
- (a) is trivial      (b) has four elements.      (c) has exactly two elements.      (d) None of these.
- (28) Let  $G$  be the group of symmetries of a regular pentagon. Then  $G$  has
- (a) 5 reflections and 5 rotations (including the trivial one).  
 (b) 10 reflections and 10 rotations.  
 (c) No reflections and 10 rotations.  
 (d) None of the above.

# Group Homomorphisms, Isomorphisms

- (1) For a positive integer  $n$ , let  $U(n) = \{\bar{x} : 1 \leq x \leq n, (x, n) = 1\}$  denote the group of prime residue classes modulo  $n$  under multiplication. Then
- (a)  $U(8)$  and  $U(10)$  are isomorphic groups.      (b)  $U(10)$  and  $U(12)$  are isomorphic groups.  
(c)  $U(8)$  and  $U(12)$  are isomorphic groups.      (d) None of the above.
- (2) Let  $\mathbb{Q}^* = \mathbb{Q} - 0$ ,  $\mathbb{R}^* = \mathbb{R} - 0$ ,  $\mathbb{Q}^+$  = set of positive rational numbers,  $\mathbb{R}^+$  = set of positive real numbers. Which of the following pairs of groups are isomorphic groups.
- (i)  $(\mathbb{Q}, +)$  and  $(\mathbb{Z}, +)$     (ii)  $(\mathbb{Q}, +)$  and  $(\mathbb{Q}^*, \cdot)$     (iii)  $(\mathbb{Q}, +)$  and  $(\mathbb{Q}^+, \cdot)$     (iv)  $(\mathbb{R}, +)$  and  $(\mathbb{R}^+, \cdot)$
- (a) (ii) and (iv) only      (b) (iii) and (iv) only  
(c) only (iv)      (d) None of these
- (3) Consider the homomorphism  $\phi : (\mathbb{C}^*, \cdot) \rightarrow (\mathbb{C}^*, \cdot)$  defined by  $\phi(x) = x^5$ . Let  $K = \ker \phi$ .
- (a)  $K$  is an infinite group of  $\mathbb{C}^*$ .  
(b)  $K$  is a trivial group.  
(c)  $K$  is the group of fifth roots of unity and  $|K| = 5$ .  
(d) None of the above.
- (4) Let  $G$  be a cyclic group of order 7. Let  $\phi : G \rightarrow G$  be defined by  $\phi(x) = x^4$ .
- (a)  $\phi$  is not a group homomorphism.  
(b)  $\phi$  is a group homomorphism which is not one-one.  
(c)  $\phi$  is a group homomorphism which is not onto.  
(d) None of the above.
- (5) Which of the following statements is **true**.
- (a)  $(\mathbb{Z}_4, +)$  and  $V_4$  (Klein's 4 group) are isomorphic.  
(b)  $(\mathbb{Z}_4, +)$  and  $\mu_4$  (The group of fourth roots of unity) are isomorphic.  
(c)  $V_4$  and  $\mu_4$  are isomorphic.  
(d) None of the above.
- (6) Let  $Q_8$  denote the quaternion group  $\{\pm 1, \pm i, \pm j, \pm k\}$  where  $i^2 = j^2 = -1, ij = k = -ji$  ( $Q_8 = \langle i, j \rangle$ ). Consider the map  $\phi : Q_8 \rightarrow Z_2$  be defined by  $\phi(i) = \bar{0}, \phi(j) = \bar{1}$ . Then
- (a)  $\phi$  is not a group homomorphism.  
(b)  $\phi$  is a group homomorphism and  $\ker \phi = \{1, i\}$ .  
(c)  $\phi$  is a group homomorphism and  $\ker \phi = \{\pm i\}$ .  
(d) None of the above.

- (7) Let  $U(16)$  denote the group of prime residue classes modulo 16 under multiplication. Which of the following statements are **false**?
- $\phi_1 : U(16) \rightarrow U(16)$  defined by  $\phi_1 = x^3$  is a group automorphism.
  - $\phi_2 : U(16) \rightarrow U(16)$  defined by  $\phi_2 = x^5$  is a group automorphism.
  - $\phi_3 : U(16) \rightarrow U(16)$  defined by  $\phi_3 = x^9$  is not a group automorphism.
  - $\phi_4 : U(16) \rightarrow U(16)$  defined by  $\phi_4 = x^4$  is not a group automorphism.
- (8) Let  $G$  be an abelian group which has no element of order 2 and  $\phi : G \rightarrow G$  is defined by  $\phi(x) = x^2$ . Then
- $\phi$  is an automorphism of  $G$ .
  - $\phi$  is a group homomorphism which may not be one-one.
  - $\phi$  is an automorphism of  $G$  if  $G$  is finite.
  - $\phi$  is not a group homomorphism.
- (9) Consider the group  $G$ , where  $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R}, a \neq 0 \right\}$  under multiplication of  $2 \times 2$  matrices. Then
- $G$  is isomorphic to  $(\mathbb{R}, +)$ .
  - $G$  is isomorphic to  $(\mathbb{R}^*, \cdot)$ .
  - $G$  is isomorphic to  $SL_2(\mathbb{R})$ .
  - $G$  is isomorphic to  $O_2(\mathbb{R})$ .
- (10) Let  $m$  and  $n$  be integers. Then
- The groups  $(m\mathbb{Z}, +)$  and  $(n\mathbb{Z}, +)$  are isomorphic.
  - The groups  $(m\mathbb{Z}, +)$  and  $(n\mathbb{Z}, +)$  are isomorphic if and only if  $m = -n$ .
  - The groups  $(m\mathbb{Z}, +)$  and  $(n\mathbb{Z}, +)$  are not isomorphic if  $m \neq n$ .
  - The groups  $(m\mathbb{Z}, +)$  and  $(n\mathbb{Z}, +)$  are isomorphic for all non-zero integers  $m$  and  $n$ .
- (11) Let  $G$  be a cyclic group of order  $n$ . Then  $Aut(G)$  has
- $n$  elements
  - $\phi(n)$  elements
  - 1 element
  - $n - 1$  elements.
- (12) The map  $f : GL_2(\mathbb{R}) \rightarrow GL_2(\mathbb{R})$  defined by  $f(A) = (A^t)^{-1}$  is
- not a group homomorphism.
  - group homomorphism and  $\ker f = SL_2\mathbb{R}$ .
  - group homomorphism and  $\ker f = O_2\mathbb{R}$ .
  - a group automorphism.
- (13) The number of group homomorphisms from  $S_3$  to  $(\mathbb{Z}_3, +)$  is
- 1
  - 0
  - 3
  - 2
- (14) The number of group automorphisms of  $V_4$  (Klein's four group) is
- 4
  - 2
  - 3
  - 6
- (15) Let  $G$  be an abelian group of order  $n$ . The map  $\phi : G \rightarrow G$  defined by  $\phi(x) = x^m$  where  $m$  is a positive integer is
- a group homomorphism if and only if  $m, n$  are relatively prime.